# Explicit Reconstruction of Homogeneous Isolated Hypersurface Singularities from their Milnor Algebras*

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*By the Mather-Yau theorem, a complex hypersurface germ $\mathcal{V}$ with isolated singularity is completely determined by its moduli algebra $\mathcal{A}(\mathcal{V})$. The proof of the theorem does not provide an explicit procedure for recovering $\mathcal{V}$ from $\mathcal{A}(\mathcal{V})$, and finding such a procedure is a long-standing open problem. In this paper, we present an explicit way for reconstructing $\mathcal{V}$ from $\mathcal{A}(\mathcal{V})$ up to biholomorphic equivalence under the assumption that the singularity of $\mathcal{V}$ is homogeneous, in which case $\mathcal{A}(\mathcal{V})$ coincides with the Milnor algebra of $\mathcal{V}$.*

## 1 Introduction

Let $\mathcal{O}_n$ be the local algebra of holomorphic function germs at the origin in $\mathbb{C}^n$ with $n \geq 2$. For every hypersurface germ $\mathcal{V}$ at the origin (considered with its reduced complex structure) denote by $I(\mathcal{V})$ the ideal of elements of $\mathcal{O}_n$ vanishing on $\mathcal{V}$. Let $f$ be a generator of $I(\mathcal{V})$, and consider the complex associative commutative algebra $\mathcal{A}(\mathcal{V})$ defined as the quotient of $\mathcal{O}_n$ by the ideal generated by $f$ and all its first-order partial derivatives. The algebra $\mathcal{A}(\mathcal{V})$, called the *moduli algebra* or *Tjurina algebra* of $\mathcal{V}$, is independent of the choice of $f$ as well as the coordinate system near the origin, and the moduli algebras of biholomorphically equivalent hypersurface germs are isomorphic. Clearly, $\mathcal{A}(\mathcal{V})$ is trivial if and only if $\mathcal{V}$ is non-singular. Furthermore, it is well-known that $0 < \dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) < \infty$ if and only if the germ $\mathcal{V}$ has an isolated singularity (see, e.g. Chapter 1 in [GLS]).

By a theorem due to Mather and Yau (see [MY]), two hypersurface germs $\mathcal{V}_1$, $\mathcal{V}_2$ in $\mathbb{C}^n$ with isolated singularities are biholomorphically equivalent if their moduli algebras $\mathcal{A}(\mathcal{V}_1)$, $\mathcal{A}(\mathcal{V}_2)$ are isomorphic. Thus, given the dimension $n$, the moduli algebra $\mathcal{A}(\mathcal{V})$ determines $\mathcal{V}$ up to biholomorphism. In particular, if $\dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) = 1$, then $\mathcal{V}$ is biholomorphic to the germ of the hypersurface $\{z_1^2 + \ldots + z_n^2 = 0\}$, and if $\dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) = 2$, then $\mathcal{V}$ is biholomorphic to the germ of the hypersurface $\{z_1^2 + \ldots + z_{n-1}^2 + z_n^3 = 0\}$. The proof of the Mather-Yau theorem does not provide an explicit procedure for recovering the germ $\mathcal{V}$ from the algebra $\mathcal{A}(\mathcal{V})$ in general, and finding a way

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for reconstructing $\mathcal{V}$ (or at least some invariants of $\mathcal{V}$) from $\mathcal{A}(\mathcal{V})$ is an interesting open problem (cf. [Y1], [Y2], [Sch], [EI]). In this paper we present an explicit method for restoring $\mathcal{V}$ from $\mathcal{A}(\mathcal{V})$ up to biholomorphic equivalence under the assumption that the singularity of $\mathcal{V}$ is homogeneous.

Let $\mathcal{V}$ be a hypersurface germ having an isolated singularity. The singularity of $\mathcal{V}$ is said to be *homogeneous* if for some (hence for every) generator $f$ of $I(\mathcal{V})$ there is a coordinate system near the origin in which $f$ becomes the germ of a homogeneous polynomial. In this case $f$ lies in the Jacobian ideal $\mathcal{J}(f)$ in $\mathcal{O}_n$, which is the ideal generated by all first-order partial derivatives of $f$. Hence, for a homogeneous singularity, $\mathcal{A}(\mathcal{V})$ coincides with the *Milnor algebra* $\mathcal{O}_n/\mathcal{J}(f)$ for any generator $f$ of $I(\mathcal{V})$.

Let $Q(z)$, with $z := (z_1, \dots, z_n)$, be a holomorphic $(m+1)$-form on $\mathbb{C}^n$, i.e. a homogeneous polynomial of degree $m+1$ in the variables $z_1, \dots, z_n$, where $m \geq 2$. Consider the germ $\mathcal{V}$ of the hypersurface $\{Q(z) = 0\}$ and assume that: (i) the singularity of $\mathcal{V}$ is isolated, and (ii) the germ of $Q$ generates $I(\mathcal{V})$. These two conditions are equivalent to the non-vanishing of the discriminant $\Delta(Q)$ of $Q$ (see Chapter 13 in [GKZ]). Next, consider the gradient map $\mathbf{Q} : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto \operatorname{grad} Q(z)$. Since $\Delta(Q) \neq 0$, the fiber $\mathbf{Q}^{-1}(0)$ consists of 0 alone; in particular, the map $\mathbf{Q}$ is finite at the origin. The main content of this paper is a procedure for recovering $\mathbf{Q}$ from $\mathcal{A}(\mathcal{V})$ up to linear equivalence, where we say that two maps $\Phi_1, \Phi_2 : \mathbb{C}^n \to \mathbb{C}^n$ are linearly equivalent if there exist non-degenerate linear transformations $L_1, L_2$ of $\mathbb{C}^n$ such that $\Phi_2 = L_1 \circ \Phi_1 \circ L_2$.

In fact, we consider a more general situation. Let $p_r$, $r = 1, \dots, n$, be holomorphic $m$-forms on $\mathbb{C}^n$ and $I$ the ideal in $\mathcal{O}_n$ generated by the germs of these forms at the origin. Define $\mathcal{A} := \mathcal{O}_n/I$ and assume that $\dim_{\mathbb{C}} \mathcal{A} < \infty$, which is equivalent to the finiteness of the map $\mathbf{P} : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto (p_1(z), \dots, p_n(z))$ at the origin (see Chapter 1 in [GLS]). Observe that since the components of $\mathbf{P}$ are homogeneous polynomials, $\mathbf{P}$ is finite at 0 if and only if $\mathbf{P}^{-1}(0) = \{0\}$. In this paper we propose a procedure (which requires only linear-algebraic manipulations) for explicitly recovering the map $\mathbf{P}$ from $\mathcal{A}$ up to linear equivalence. As explained in Remark 2.1, this procedure also helps decide whether a given complex finite-dimensional associative algebra is isomorphic to an algebra arising from a finite homogeneous polynomial map as above.

The paper is organized as follows. Reconstruction of $\mathbf{P}$ from $\mathcal{A}$ is done in Section 2. In Section 3 we apply our method to the algebra $\mathcal{A}(\mathcal{V})$ arising from $Q$ to obtain a map $\mathbf{Q}'$ linearly equivalent to $\mathbf{Q}$. It is then not hard to derive from $\mathbf{Q}'$ an $(m+1)$-form $Q'$ linearly equivalent to $Q$, where two forms $Q_1$, $Q_2$ on $\mathbb{C}^n$ are called linearly equivalent if there exists a non-degenerate linear transformation $L$ of $\mathbb{C}^n$ such that $Q_2 = Q_1 \circ L$. Then the germ of the hypersurface $\{Q'(z) = 0\}$ is the sought-after reconstruction of $\mathcal{V}$ from $\mathcal{A}(\mathcal{V})$ up to biholomorphic equivalence. We conclude the paper by illustrating our

reconstruction procedure with the example of simple elliptic singularities of type $\widetilde{E}_6$.

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## 2 Reconstruction of finite polynomial maps

Recapping the setup outlined in the introduction, let $p_r$, $r = 1, \dots, n$, be holomorphic $m$-forms on $\mathbb{C}^n$ and $I$ the ideal in $\mathcal{O}_n$ generated by the germs of these forms at the origin, where $m, n \geq 2$. Define $\mathcal{A} := \mathcal{O}_n/I$ and assume that $\dim_{\mathbb{C}} \mathcal{A} < \infty$ (observe that $\dim_{\mathbb{C}} \mathcal{A} \geq m+1$). In this section we present a method for recovering the map $\mathbf{P} : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto (p_1(z), \dots, p_n(z))$ from $\mathcal{A}$ up to linear equivalence. Everywhere below we suppose that $\mathcal{A}$ is given as an abstract associative algebra, i.e. by a multiplication table with respect to some basis $e_1, \dots, e_N$, with $N := \dim_{\mathbb{C}} \mathcal{A}$.

First of all, we find the unit $\mathbf{1}$ of $\mathcal{A}$. One has $\mathbf{1} = \sum_{k=1}^N \alpha_k e_k$ where the coefficients $\alpha_k \in \mathbb{C}$ are uniquely determined from the linear system

$$\sum_{k=1}^{N} \alpha_k (e_k e_\ell) = e_\ell, \quad \ell = 1, \dots, N.$$

Assume now that $e_1 = \mathbf{1}$ and find the maximal ideal $\mathfrak{m}$ of $\mathcal{A}$. Clearly, $\mathfrak{m}$ is spanned by the vectors $e'_k := e_k - \beta_k \mathbf{1}$, $k = 2, \dots, N$, where $\beta_k \in \mathbb{C}$ are uniquely fixed by the requirement that each $e'_k$ is not invertible in $\mathcal{A}$. Hence, for each $k$ the number $\beta_k$ is determined from the condition that the linear system

$$\sum_{\ell=1}^{N} \gamma_\ell (e_k e_\ell - \beta_k e_\ell) = e_1 \tag{2.1}$$

cannot be solved for $\gamma_1, \dots, \gamma_N \in \mathbb{C}$. Since system (2.1) has at most one solution for any $\beta_k$, this condition is equivalent to the degeneracy of the coefficient matrix $M_k$ of (2.1). We have $M_k = C_k - \beta_k \mathrm{Id}$, where $C_k := (c_{kj\ell})_{j,\ell=1,\dots,N}$, with $c_{kj\ell}$ given by $e_k e_\ell = \sum_{j=1}^N c_{kj\ell} e_j$. It then follows that the required value of $\beta_k$ is the (unique) eigenvalue of the matrix $C_k$.

We are now in a position to find the number of variables $n$ and the degree $m$ for the forms $p_r$. By Nakayama's lemma, $\mathfrak{m}$ is a nilpotent algebra, and we denote by $\nu$ its nil-index, which is the largest integer $\mu$ with $\mathfrak{m}^\mu \neq 0$. Observe that $\nu \leq N-1$, and therefore to determine $\nu$ it is sufficient to compute all products of the basis vectors $e'_k$ of length not exceeding $N-1$. Further, since $\mathcal{A}$ is finite-dimensional, the forms $p_r$ form a regular sequence in $\mathcal{O}_n$ (see Theorem 2.1.2 in [BH]). Hence $\mathcal{A}$ is a complete intersection ring, which implies that $\mathcal{A}$ is a Gorenstein algebra (see [B]). Recall that a

(complex) local commutative associative algebra $\mathcal{B}$ with $1 < \dim_{\mathbb{C}} \mathcal{B} < \infty$ is Gorenstein if and only if for the annihilator $\mathrm{Ann}(\mathfrak{n}) := \{x \in \mathfrak{n} : x\,\mathfrak{n} = 0\}$ of its maximal ideal $\mathfrak{n}$ one has $\dim_{\mathbb{C}} \mathrm{Ann}(\mathfrak{n}) = 1$ (see e.g. [H]). Lemma 3.4 of [Sa] yields that $\mathrm{Ann}(\mathfrak{m})$ is spanned by the element represented by the germ of $J(\mathbf{P}) := \det(\partial p_r/\partial z_s)_{r,s=1,\dots,n}$.

For every $i > 0$, let $\mathcal{P}_i$ be the vector space of all $i$-forms on $\mathbb{C}^n$ and $\mathcal{L}_i$ the linear subspace of $\mathcal{A}$ that consists of all elements represented by germs of forms in $\mathcal{P}_i$. Since $\mathfrak{m}$ consists of all elements of $\mathcal{A}$ represented by germs in $\mathcal{O}_n$ vanishing at the origin, the subspaces $\mathcal{L}_i$ lie in $\mathfrak{m}$ and yield a grading on $\mathfrak{m}$:

$$\mathfrak{m} = \bigoplus_{i>0} \mathcal{L}_i, \quad \mathcal{L}_i \mathcal{L}_j \subset \mathcal{L}_{i+j} \text{ for all } i, j.$$

Since $\dim_{\mathbb{C}} \mathrm{Ann}(\mathfrak{m}) = 1$, it immediately follows that $\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}^{\nu} = \mathcal{L}_d$ for $d := \max\{i : \mathcal{L}_i \neq 0\}$. On the other hand, $\mathrm{Ann}(\mathfrak{m})$ is spanned by the element represented by the germ of $J(\mathbf{P})$, which is an $n(m-1)$-form. Thus $d = n(m-1)$. Furthermore, we have

$$\begin{aligned} &\dim_{\mathbb{C}} \mathcal{L}_i = \dim_{\mathbb{C}} \mathcal{P}_i \quad \text{for } i = 1, \dots, m-1, \\ &\dim_{\mathbb{C}} \mathcal{L}_m = \dim_{\mathbb{C}} \mathcal{P}_m - n. \end{aligned} \tag{2.2}$$

Now, observe that

$$\mathcal{L}_i = \mathcal{L}_1^i \text{ for all } i, \tag{2.3}$$

i.e. the graded algebra $\mathcal{A}$ is *standard* in the terminology of [St]. Hence $\mathcal{L}_i$ is a complement to $\mathfrak{m}^{i+1}$ in $\mathfrak{m}^i$ for all $i > 0$. For $i = 1$ this implies that $n$ can be recovered from the algebra $\mathcal{A}$ as

$$n = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \tag{2.4}$$

(see Remark 3.2). Next, for $i = \nu$ we obtain $n(m-1) = \nu$ (in particular, $n$ divides $\nu$). Thus, the degree $m$ of the forms $p_r$ can be recovered from $\mathcal{A}$ as follows:

$$m = \nu/n + 1. \tag{2.5}$$

Note that since $\mathcal{A}$ is given as an abstract associative algebra, finding the grading $\{\mathcal{L}_i\}$ from the available data may be hard. We stress that determination of this grading is not required for recovering $n$ and $m$.

Further, choose an arbitrary basis $f_1, \dots, f_n$ in a complement to $\mathfrak{m}^2$ in $\mathfrak{m}$. Clearly, for some $C \in \mathrm{GL}(n, \mathbb{C})$ one has

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = C \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix},$$

where $Z_j$ is the element of $\mathcal{L}_1$ represented by the germ of the coordinate function $z_j$, and $W_j \in \mathfrak{m}^2$. Set

$$K := \dim_{\mathbb{C}} \mathcal{P}_m = \left( \begin{array}{c} m+n-1 \\ m \end{array} \right),$$

and let $q_1(z), \dots, q_K(z)$ be the monomial basis of $\mathcal{P}_m$ where $z := (z_1, \dots, z_n)$. Next, fix a complement $\mathcal{S}$ to $\mathfrak{m}^{m+1}$ in $\mathfrak{m}^m$ and let $\pi : \mathfrak{m}^m \to \mathcal{S}$ be the projection onto $\mathcal{S}$ with kernel $\mathfrak{m}^{m+1}$. Condition (2.3) for $i = m$ then yields that $\mathcal{S}$ is spanned by $\pi(q_1(f)), \dots, \pi(q_K(f))$, where $f := (f_1, \dots, f_n)$. On the other hand, by (2.2) we have $\dim_{\mathbb{C}} \mathcal{S} = K - n$. Hence one can find $n$ linear relations

$$\sum_{\rho=1}^{K} \gamma_{\sigma\rho} \pi(q_\rho(f)) = 0, \quad \sigma = 1, \dots, n, \tag{2.6}$$

where the vectors $\gamma_\sigma := (\gamma_{\sigma\,1}, \dots, \gamma_{\sigma\,K}) \in \mathbb{C}^K$ are linearly independent.

Further, extracting from (2.6) the $\mathcal{L}_m$-components one obtains

$$\sum_{\rho=1}^{K} \gamma_{\sigma\rho} q_\rho(CZ) = 0, \quad \sigma = 1, \dots, n, \tag{2.7}$$

where $Z := (Z_1, \dots, Z_n)$. Identity (2.7) is equivalent to

$$\sum_{\rho=1}^{K} \gamma_{\sigma\rho} q_\rho(Cz) \in I, \quad \sigma = 1, \dots, n, \tag{2.8}$$

where each $q_\rho(z)$ is identified with its germ at the origin. From (2.8) one immediately obtains that for some matrix $D \in \mathrm{GL}(n, \mathbb{C})$ the following holds:

$$\Gamma q(Cz) \equiv D\mathbf{P}(z),$$

where $\Gamma := (\gamma_{\sigma\rho})_{\sigma=1,\dots,n,\, \rho=1,\dots,K,}$ and $q := (q_1, \dots, q_K)$. Thus, the map

$$\Phi : \mathbb{C}^n \to \mathbb{C}^n, \quad z \mapsto \Gamma q(z) \tag{2.9}$$

is linearly equivalent to $\mathbf{P}$ as required.

We now summarize the main steps of our algorithm for recovering $\mathbf{P}$ from

$\mathcal{A}$ up to linear equivalence:

1. Find $\mathfrak{m}$ and its nil-index $\nu$.
2. Determine $n$ from formula (2.4).
3. Determine $m$ from formula (2.5).
4. Choose a complement to $\mathfrak{m}^2$ in $\mathfrak{m}$ and an arbitrary basis $f_1, \dots, f_n$ in this complement.
5. Calculate $q_1(f), \dots, q_K(f)$, where $f := (f_1, \dots, f_n)$ and $q_1(z), \dots, q_K(z)$ are all monomials of degree $m$ in $z := (z_1, \dots, z_n)$.
6. Choose a complement $\mathcal{S}$ to $\mathfrak{m}^{m+1}$ in $\mathfrak{m}^m$.
7. Compute $\pi(q_1(f)), \dots, \pi(q_K(f))$, where $\pi : \mathfrak{m}^m \to \mathcal{S}$ is the projection onto $\mathcal{S}$ with kernel $\mathfrak{m}^{m+1}$.
8. Find $n$ linearly independent linear relations among the vectors $\pi(q_1(f)), \dots, \pi(q_K(f))$ as in (2.6).
9. Formula (2.9) then gives a map linearly equivalent to $\mathbf{P}$.

In the next section this algorithm will be applied to the gradient map arising from a form with non-zero discriminant.

**Remark 2.1** A natural problem is to characterize the algebras that arise from finite homogeneous polynomial maps as above among all complex finite-dimensional Gorenstein algebras. This problem is a special case of the well-known *recognition problem* for the moduli algebras of general isolated hypersurface singularities and the corresponding Lie algebras of derivations (see, e.g. [Y1], [Y2], [Sch]). The algorithm presented here can help decide whether a given finite-dimensional Gorenstein algebra $\mathcal{B}$ is isomorphic to an algebra $\mathcal{A}$ of the kind considered in this section. Indeed, one can attempt to formally apply the algorithm to $\mathcal{B}$. For the algorithm to go through one requires that: (i) the nil-index of the maximal ideal of $\mathcal{B}$ be divisible by the number $n$ found from formula (2.4), (ii) for some basis $f_1, \dots, f_n$ in some complement to $\mathfrak{m}^2$ in $\mathfrak{m}$ and for some complement $\mathcal{S}$ to $\mathfrak{m}^{m+1}$ in $\mathfrak{m}^m$ there exist $n$ linearly independent linear relations among the vectors $\pi(q_1(f)), \dots, \pi(q_K(f))$, with $m$ being the number found from formula (2.5), and (iii) the map $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ produced on Step 9 be finite at the origin. If the algorithm fails (i.e. some of conditions (i)–(iii) are not satisfied), then $\mathcal{B}$ does not arise from a finite homogeneous polynomial map. If the algorithm successfully finishes, the resulting map $\Phi$ is a candidate map from which $\mathcal{B}$ may potentially arise. In order to see if this is indeed the case, one needs to check whether $\mathcal{B}$ is isomorphic to the algebra associated to $\Phi$. For this purpose one can use the criterion for isomorphism of finite-dimensional Gorenstein algebras established in [FIKK].

# 3 Reconstruction of homogeneous singularities

Suppose now that $\mathbf{P} = \mathbf{Q} := \operatorname{grad} Q$ for a holomorphic $(m+1)$-form $Q$ on $\mathbb{C}^n$ with $\Delta(Q) \neq 0$, where $\Delta$ is the discriminant. Let $\Phi$ be a map linearly equivalent to $\mathbf{Q}$ produced by the procedure described in Section 2 from the algebra $\mathcal{A}(\mathcal{V})$, where $\mathcal{V}$ is the germ of the hypersurface $\{Q = 0\}$ at the origin. We then have

$$\Phi(z) \equiv C_1 \operatorname{grad} Q(C_2 z) \tag{3.1}$$

for some $C_1, C_2 \in \mathrm{GL}(n, \mathbb{C})$. Our next task is to recover $Q$ from $\Phi$ up to linear equivalence.

Let $Q'$ be the $(m+1)$-form defined by $Q'(z) := Q(C_2 z)$ for all $z \in \mathbb{C}^n$. Then $\operatorname{grad} Q(C_2 z) = (C_2^{-1})^T \operatorname{grad} Q'(z)$, and (3.1) implies

$$\Phi(z) \equiv C \operatorname{grad} Q'(z)$$

for some $C \in \mathrm{GL}(n, \mathbb{C})$. For any $n \times n$-matrix $D$ we now introduce the holomorphic differential 1-form $\omega^D := \sum_{r=1}^n \Psi_r^D dz_r$ on $\mathbb{C}^n$, where $(\Psi_1^D, \dots, \Psi_n^D)$ are the components of the map $\Psi^D := D\,\Phi$. Consider the equation

$$d\omega^D = 0 \tag{3.2}$$

as a linear system with respect to the entries of the matrix $D$. Clearly, $C^{-1}$ is a solution of (3.2). Let $D_0$ be another solution of (3.2) and assume that $D_0 \in \mathrm{GL}(n, \mathbb{C})$. Every closed holomorphic differential form on $\mathbb{C}^n$ is exact, and integrating $\Psi^{D_0}$ one obtains an $(m+1)$-form $Q''$ on $\mathbb{C}^n$. Then $\operatorname{grad} Q'' = \Psi^{D_0} = D_0 C \operatorname{grad} Q'$, and therefore $\Delta(Q'') \neq 0$. Furthermore, the Milnor algebras of the germs $\mathcal{V}'$ and $\mathcal{V}''$ of the hypersurfaces $\{Q'(z) = 0\}$ and $\{Q''(z) = 0\}$ coincide. By the Mather-Yau theorem, this implies that $\mathcal{V}'$ and $\mathcal{V}''$ are biholomorphically equivalent and therefore $\mathcal{V}''$ is biholomorphically equivalent to $\mathcal{V}$, which yields that $Q''$ is linearly equivalent to $Q$. Thus, any non-degenerate matrix that solves linear system (3.2) leads to an $(m+1)$-form linearly equivalent to $Q$ and a hypersurface germ biholomorphically equivalent to $\mathcal{V}$.

We will now illustrate our method for recovering $\mathcal{V}$ from $\mathcal{A}(\mathcal{V})$ by the example of simple elliptic singularities of type $\widetilde{E}_6$. These singularities form a family parametrized by $t \in \mathbb{C}$ satisfying $t^3 + 27 \neq 0$. Namely, for every such $t$ let $\mathcal{V}_t$ be the germ at the origin of the hypersurface $\{Q_t(z) = 0\}$, where $Q_t$ is the following cubic on $\mathbb{C}^3$:

$$Q_t(z) := z_1^3 + z_2^3 + z_3^3 + t z_1 z_2 z_3, \quad z := (z_1, z_2, z_3).$$

Below we explicitly show how $Q_t$ can be recovered from the algebra $\mathcal{A}_t := \mathcal{A}(\mathcal{V}_t)$ up to linear equivalence.

Recall that the starting point of our reconstruction procedure is a multiplication table with respect to some basis. The algebra $\mathcal{A}_t$ has dimension 8 and with respect to a certain basis $e_1, \dots, e_8$ is given by (see Remark 3.1 below):

$$
\begin{aligned}
&e_1e_j = e_j \text{ for } j = 1, \dots, 8,\ e_2^2 = -\frac{t}{3}e_3 + \frac{2t}{3}e_6,\ e_2e_3 = e_6,\\
&e_2e_4 = e_5 - e_6 - e_8,\ e_2e_5 = e_7,\ e_2e_6 = 0,\ e_2e_7 = 0,\ e_2e_8 = e_7,\\
&e_3e_j = 0 \text{ for } j = 3, \dots, 8,\ e_4^2 = -\frac{t}{3}e_7,\ e_4e_5 = e_3 - 2e_6,\ e_4e_6 = 0,\\
&e_4e_7 = e_6,\ e_4e_8 = e_3 - 2e_6,\ e_5^2 = -\frac{t}{3}e_5 + (2+t)e_6 + \frac{t}{3}e_8,\ e_5e_6 = 0,\\
&e_5e_7 = 0,\ e_5e_8 = -\frac{t}{3}e_5 + (1+t)e_6 + \frac{t}{3}e_8,\ e_6e_j = 0 \text{ for } j = 6, 7, 8,\\
&e_7e_j = 0 \text{ for } j = 7, 8,\ e_8^2 = -\frac{t}{3}e_5 + te_6 + \frac{t}{3}e_8.
\end{aligned}
\tag{3.3}
$$

It is clear from (3.3) that $e_1 = \mathbf{1}$ and $\mathfrak{m}_t = \langle e_2, \dots, e_8\rangle$, where $\langle\,\cdot\,\rangle$ denotes linear span and $\mathfrak{m}_t$ is the maximal ideal of $\mathcal{A}_t$. We then have $\mathfrak{m}_t^2 = \langle e_3, e_6, e_7, e_5 - e_8\rangle$, $\mathfrak{m}_t^3 = \langle e_6\rangle$, $\mathfrak{m}_t^4 = 0$, hence $\nu = 3$. Further, by formula (2.4) we obtain $n = 3$, which together with formula (2.5) yields $m = 2$.

We now list all monomials of degree 2 in $z$ as follows:

$$q_1(z) := z_1^2,\ q_2(z) := z_2^2,\ q_3(z) := z_3^2,\ q_4(z) := z_1z_2,\ q_5(z) := z_1z_3,\ q_6(z) := z_2z_3$$

(here $K = 6$). Next, we let $f_1 := e_2$, $f_2 := e_4$, $f_3 := e_5$, which for $f := (f_1, f_2, f_3)$ yields

$$
\begin{aligned}
&q_1(f) = -\frac{t}{3}e_3 + \frac{2t}{3}e_6,\ q_2(f) = -\frac{t}{3}e_7,\ q_3(f) = -\frac{t}{3}e_5 + (2+t)e_6 + \frac{t}{3}e_8,\\
&q_4(f) = e_5 - e_6 - e_8, \quad q_5(f) = e_7, \quad q_6(f) = e_3 - 2e_6.
\end{aligned}
$$

Further, define $\mathcal{S} := \langle e_3, e_7, e_5 - e_8\rangle$. Clearly, $\mathcal{S}$ is a complement to $\mathfrak{m}_t^3$ in $\mathfrak{m}_t^2$. Then for the projection $\pi : \mathfrak{m}_t^2 \to \mathcal{S}$ with kernel $\mathfrak{m}_t^3$ one has

$$
\begin{aligned}
&\pi(q_1(f)) = -\frac{t}{3}e_3,\ \pi(q_2(f)) = -\frac{t}{3}e_7,\ \pi(q_3(f)) = -\frac{t}{3}e_5 + \frac{t}{3}e_8,\\
&\pi(q_4(f)) = e_5 - e_8, \quad \pi(q_5(f)) = e_7, \quad \pi(q_6(f)) = e_3.
\end{aligned}
$$

The vectors $\pi(q_1(f)), \dots, \pi(q_6(f))$ satisfy the following three linearly independent linear relations:

$$\pi(q_1(f)) + \frac{t}{3}\pi(q_6(f)) = 0,\ \pi(q_2(f)) + \frac{t}{3}\pi(q_5(f)) = 0,\ \pi(q_3(f)) + \frac{t}{3}\pi(q_4(f)) = 0.$$

Hence we have

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t/3 \\ 0 & 1 & 0 & 0 & t/3 & 0 \\ 0 & 0 & 1 & t/3 & 0 & 0 \end{pmatrix},$$

which for $q(z) := (q_1(z), \ldots, q_6(z))$ yields

$$\Phi(z) = \Gamma q(z) = \begin{pmatrix} z_1^2 + \dfrac{t}{3} z_2 z_3 \\[2ex] z_2^2 + \dfrac{t}{3} z_1 z_3 \\[2ex] z_3^2 + \dfrac{t}{3} z_1 z_2 \end{pmatrix}.$$

It remains to recover $Q_t$ from $\Phi$ up to linear equivalence. For

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

system (3.2) is equivalent to the following system of equations:

$$\begin{aligned} &2d_{12} - \frac{t}{3} d_{23} = 0, \quad 2d_{21} - \frac{t}{3} d_{13} = 0, \quad t d_{11} - t d_{22} = 0, \\ &2d_{13} - \frac{t}{3} d_{32} = 0, \quad 2d_{31} - \frac{t}{3} d_{12} = 0, \quad t d_{11} - t d_{33} = 0, \\ &2d_{23} - \frac{t}{3} d_{31} = 0, \quad 2d_{32} - \frac{t}{3} d_{21} = 0, \quad t d_{22} - t d_{33} = 0. \end{aligned} \tag{3.4}$$

If $t \neq 0$ and $t^3 \neq 216$, the only non-degenerate solutions of (3.4) are non-zero scalar matrices. Integrating $\Psi^D$ for such a matrix $D$ we obtain a form proportional to $Q_t$, which is obviously linearly equivalent to $Q_t$. If $t = 0$, any non-degenerate solution of (3.4) is a diagonal matrix with non-zero $d_{11}$, $d_{22}$, $d_{33}$. Integrating $\Psi^D$ for such a matrix $D$ we obtain the form

$$\frac{1}{3}\left(d_{11} z_1^3 + d_{22} z_2^3 + d_{33} z_3^3\right),$$

which is linearly equivalent to $Q_0 = z_1^3 + z_2^3 + z_3^3$ by suitable dilations of the variables.

The remaining case $t^3 = 216$ is more interesting. Writing $t = 6\lambda$ with $\lambda^3 = 1$, we see that $D$ is a solution of (3.4) if and only if

$$d_{11} = d_{22} = d_{33},\ d_{12} = \lambda^2 d_{31},\ d_{23} = \lambda d_{31},\ d_{21} = \lambda^2 d_{32},\ d_{13} = \lambda d_{32}.$$

Such a matrix $D$ is non-degenerate if and only if $d_{11}^3+d_{31}^3+d_{32}^3-3\lambda d_{11}d_{31}d_{32}\neq 0$. For example, letting $d_{11}=0$, $d_{31}=0$, $d_{32}=1$ one obtains

$$\Psi^D=\begin{pmatrix}\lambda z_3^2+2\lambda^2 z_1z_2\\ \\ \lambda^2 z_1^2+2z_2z_3\\ \\ z_2^2+2\lambda z_1z_3\end{pmatrix}.$$

Integration of $\Psi^D$ leads to the form $\mathcal{Q}_\lambda:=\lambda^2 z_1^2z_2+\lambda z_1z_3^2+z_2^2z_3$. As we have noted above, the Mather-Yau theorem implies that $\mathcal{Q}_\lambda$ is linearly equivalent to $Q_t$. Furthermore, the cubic $\mathcal{Q}_\lambda$ is equivalent to $\mathcal{Q}_1$ by the map $(z_1,z_2,z_3)\mapsto(z_1/\lambda,z_2,z_3)$. Hence each of the three cubics $\mathcal{Q}_\lambda$ with $\lambda^3=1$ is linearly equivalent to each of the three cubics $Q_t$ with $t^3=216$.

This last fact can also be understood without referring to the Mather-Yau theorem, as follows. It is well-known that all non-equivalent ternary cubics with non-vanishing discriminant are distinguished by the invariant

$$\mathtt{J}:=\frac{\mathtt{I}_4^3}{\Delta},$$

where $\mathtt{I}_4$ is a certain classical $\mathrm{SL}(3,\mathbb{C})$-invariant of degree 4 (see, e.g. pp. 381–389 in [El]). For any ternary cubic $Q$ with $\Delta(Q)\neq 0$ one has $\mathtt{J}(Q)=j(Z_Q)/110592$ where $j(Z_Q)$ is the value of the $j$-invariant for the elliptic curve $Z_Q$ in $\mathbb{CP}^2$ defined by $Q$. Details on computing $\mathtt{J}(Q)$ for any $Q$ can be found, for example, in [Ea]. In particular, $\mathtt{J}(\mathcal{Q}_1)=0$ and for the cubic $Q_t$ with any $t\in\mathbb{C}$, $t^3+27\neq 0$, one has

$$\mathtt{J}(Q_t)=-\frac{t^3(t^3-216)^3}{110592(t^3+27)^3}.$$

It then follows that each of the cubics $\mathcal{Q}_\lambda$ with $\lambda^3=1$ is linearly equivalent to each of the cubics $Q_t$ with $t^3=216$, as stated above.

**Remark 3.1** One basis in which the algebra $\mathcal{A}_t$ is given by multiplication table (3.3) is as follows:

$$e_1=\mathbf{1},\ e_2=Z_1+Z_1Z_3,\ e_3=Z_2Z_3+2Z_1Z_2Z_3,\ e_4=Z_2+Z_2Z_3,$$

$$e_5=Z_3+Z_1Z_2+3Z_1Z_2Z_3,\ e_6=Z_1Z_2Z_3,\ e_7=Z_1Z_3,\ e_8=Z_3.$$

Note that in our reconstruction of $Q_t$ from $\mathcal{A}_t$ above we only used table (3.3), not the explicit form of the basis.

**Remark 3.2** As was noted in the introduction, a hypersurface germ $\mathcal{V}$ with isolated singularity is determined, in general, by the algebra $\mathcal{A}(\mathcal{V})$ and the dimension $n$ of the ambient space. We stress that in the case of homogeneous singularities the dimension $n$ can be extracted from $\mathcal{A}(\mathcal{V})$ (see formula (2.4)).

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